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Wave Probabilistic Model of Binary Time Series

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This paper presents the theory of wave probabilistic models, together with important features, such as the inclusion-exclusion rule, the product rule, the complementary principle and entanglement. These features are mathematically described, and an illustrative example of binary time series is shown to demonstrate possible applications of the theory.

Keywords: quantum models, wave probabilistic functions, binary time series.

1 Introduction

The problem of complex probability functions is gaining in importance in quantum mechanics [1, 9], where the phase functions has been recognized as necessary for information processing and quantum system modeling. The contextual interpretation of phase functions was presented in [7], and the wave probabilistic models were introduced as necessary part of probabilistic multi-models. A more rigorous introduction to wave probabilistic models was presented in [14], where phase parameters are interpreted as dependency functions between events. The link between wave probabilistic functions and the complementarity principle was first introduced in [10]. The quantization principle as the consequence of phase parameters was defined in [6].

The goal of this paper is to continue in this way of thinking and to provide a more rigorous definition of wave probabilistic models together with their basic features. Chapter 2 presents the mathematical theory of wave functions, together with their geometric interpretation. Chapter 3 covers the link between wave probabilities and entanglement. Chapter 4 describes the general estimation algorithm of wave probabilities. Chapter 5 presents the methodology for modeling non-ergodic binary time series. Chapter 6 offers an illustrative example of binary time series, and Chapter 7 concludes the paper.

2 Mathematical theory of wave probabilistic functions

A probability space consists of a sample space S and a probability function P(.), mapping the events of S to real numbers in [0,1], such that P(S) = 1, and if A_1, A_2, … is a sequence of disjoint events, then the sum rule is fulfilled:

$$P\left(\bigcup_{i\in N} A_i\right) = \sum_{i\in N} P(A_i). \tag{1}$$

If the events A_1, A_2, … are not disjoint the following (product and inclusion-exclusion) rules can be defined:

$$\begin{aligned} &P(A_1 \cap A_2 \cap \ldots \cap A_N) \\ &= P(A_1)\cdot P(A_2|A_1) \\ &\cdot P(A_3|A_1 \cap A_2)\cdot P(A_N|A_1 \cap \ldots \cap A_{N-1}), \end{aligned} \tag{2}$$

$$\begin{aligned} &P(A_1 \cup A_2 \cup \ldots \cup A_N) \\ &= \sum_{i=1}^{N} P(A_i) - \sum_{i<j}^{N} P(A_i \cap A_j) \\ &+ \sum_{i<j<k}^{N} P(A_i \cap A_j \cap A_k) + \ldots \\ &+ (-1)^{N-1}\cdot P(A_1 \cap A_2 \cap \ldots \cap A_N). \end{aligned} \tag{3}$$

Taking into consideration the basic laws of probability defined above, we can rewrite them with the help of the complex representations (wave functions) summarized in Theorem 1.

Theorem 1: Let us define N events A_i, $i \in \{1, 2, \ldots, N\}$ of a sample space S, with defined probability functions $P(A_i)$, $i \in \{1, 2, \ldots, N\}$ and let us define the N complex functions

$$\psi(A_i) = \alpha_i \cdot e^{j v_i} = \sqrt{P(A_i)}\cdot e^{j v_i},\ i \in \{1, 2, \ldots, N\} \tag{4}$$

together with their superposition state $|\psi\rangle_\eta$ as a quantum object at its measurement place (or at time series position) η:

$$\begin{aligned} |\psi\rangle_\eta &= \psi(A_1)\cdot|A_1\rangle_\eta + \psi(A_2)\cdot|A_2\rangle_\eta \\ &+ \ldots + \psi(A_N)\cdot|A_N\rangle_\eta \end{aligned} \tag{5}$$

with modules $\sqrt{P(A_i)}$ and phases v_i where the reference phase assigned to event A_1 is chosen as $v_1 = 0$, then the inclusion-exclusion rule given in (3) is represented for measurements on quantum objects $\eta + 1, \ldots, \eta + N$ (or at a time series window) by:

$$P(|A_1\rangle_{\eta+1} \cup |A_2\rangle_{\eta+2} \cup \ldots \cup |A_N\rangle_{\eta+N}) = \left|\sum_{i=1}^{N} \psi(A_i)\right|^2, \tag{6}$$

where the phases v_i are given:

$$v_i = a\cos\left(-\frac{1}{2}\frac{P((A_1 \cup A_2 \cup \ldots \cup A_{i-1}) \cap A_i)}{\sqrt{P(A_1 \cup A_2 \cup \ldots \cup A_{i-1})\cdot P(A_i)}}\right) + v_{1,2,\ldots,i-1} \tag{7}$$

and $v_{1,2,\ldots,i-1}$ is computed:

$$\begin{aligned} &\psi(A_1 \cup A_2 \cup \ldots \cup A_{i-1}) = \psi(A_1) + \ldots + \psi(A_{i-1}) \\ &= \alpha_{1,2,\ldots,i-1}\cdot e^{j v_{1,2,\ldots,i-1}} \end{aligned} \tag{8}$$

Proof: The proof is presented in [15].

In the next part of this paper, the set of complex functions (4) plus superposition state (5) together with the inclusion-exclusion rule (6, 7, 8) is called the wave probabilistic model.

Theorem 2: Let us define N events A_i, $i \in \{1, 2, \ldots, N\}$ of a sample space S, with defined probability functions $P(A_i)$, $i \in \{1, 2, \ldots, N\}$, and the N complex functions $\psi(A_i)$ with modules $\sqrt{P(A_i)}$ and phases v_i defined in (4–8), where the reference phase assigned to event A_1 is chosen as $v_1 = 0$, then the inclusion-exclusion rule given for the subset of events A_r, $r \in \{k_1, k_2, \ldots, k_m\}$ is given as:

$$P(\left|A_{k_1}\right\rangle_{\eta+1} \cup \left|A_{k_2}\right\rangle_{\eta+2} \cup \ldots \cup \left|A_{k_m}\right\rangle_{\eta+m}) = \\ = \lim_{\substack{\sqrt{P(A_k)} \to 0 \\ k \neq k_1, \ldots, k_m}} \left| \sum_{i=1}^{N} \psi(A_i) \right|^2 \tag{9}$$

Proof: The proof of Theorem 2 arises directly from Theorem 1, which was proven for all probabilistic values $\sqrt{P(A_i)}$, A_i, $i \in \{1, 2, \ldots, N\}$ including zeros. The zero probabilistic values have an impact on phase parameters v_i and change them in such a way that equation (9) is fulfilled.

In this paper we assume that quantum objects η are distinguishable. When two identical particles interact (there is a significant overlap of their wave functions), we can not distinguish between them. These overlapping quantum objects are in general bosons (plus sign corresponding to a symmetric wave function under exchange of quantum objects) or fermions (minus sign corresponding to an anti-symmetric wave function under exchange of quantum objects).

3 Wave probabilities and entanglement

Quantum entanglement (the definition for describing the entanglement principle from Wikipedia, available on: http://en.wikipedia.org/wiki/Quantum_entanglement) is a quantum mechanical phenomenon in which the quantum states of two or more quantum objects have to be described with reference to each other, even though the individual objects may be spatially separated. This leads to correlations between observable physical properties of the systems. For example, it is possible to prepare two particles in a single quantum state such that when one is observed to be spin-up, the other one will always be observed to be spin-down and vice versa, despite the fact that it is impossible to predict, according to quantum mechanics, which set of measurements will be observed. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it.

Theorem 3: Let us define N events A_i, $i \in \{1, 2, \ldots, N\}$ of a sample space S, with defined probability functions $P(A_i)$, $i \in \{1, 2, \ldots, N\}$, and the N complex functions $\psi(A_i)$ with modules $\sqrt{P(A_i)}$ and phases v_i defined in (4–8), where the reference phase assigned to event A_1 is chosen as $v_1 = 0$, then all events A_r, $r \in \{r_1, r_2, \ldots, r_n\}$ with only two possible states $\left|A_{r_i}\right\rangle_{\eta+i}$ $\left|\overline{A}_{r_i}\right\rangle_{\eta+i}$ are entangled if the following form holds:

$$P(\left|A_{r_1}\right\rangle_{\eta+1} \cup \ldots \cup \left|A_{r_n}\right\rangle_{\eta+n}) = \\ = \lim_{\substack{\sqrt{P(A_k)} \to 0 \\ k \neq r_1, r_2, \ldots, r_n}} \left|\psi(A_1) + \psi(A_2) + \ldots + \psi(A_N)\right|^2 \equiv 0 \tag{10}$$

which yields into form:

$$P(\left|\overline{A}_{r_1}\right\rangle_{\eta+1} \cap \ldots \cap \left|\overline{A}_{r_n}\right\rangle_{\eta+n}) = 1 \tag{11}$$

where $\left|\overline{A}_{r_i}\right\rangle_{\eta+i}$ means the inversion state in comparison with $\left|A_{r_i}\right\rangle_{\eta+i}$ on quantum object $(\eta + i)$.

Proof: Theorem 3 emerges directly from the inclusion-exclusion rules and from the definition of wave probabilities. Phase parameters assigned into wave functions can be either positive or negative.

Form (10) modifies the phases for the selected subset of events $\{A_{r_1}, A_{r_2}, \ldots, A_{r_n}\}$ to comply with the inclusion-exclusion rule (9). For special cases, the inclusion-exclusion rule can yield into zero due to wave resonances within the phases of events. This special case can occur for a special set up of phases.

Zero probability (10) directly yields into equation (11), which defines that the state characterized by $\overline{A}_{r_1} \cap \ldots \cap \overline{A}_{r_n}$ will surely occur. This state is not random but fully deterministic and so spatially spread within events $\{A_{r_1}, A_{r_2}, \ldots, A_{r_n}\}$.

It can be stated that entanglement is the logical result of probabilistic wave functions, and represents something like the resonance wave functions yielding into deterministic states.

4 Estimation algorithm of the wave probabilistic model of time series

We assume the time series composed of many quantum objects, each of which is described by wave function covering the superposition principle of all possible events $\{A_1, A_2, \ldots, A_N\}$:

$$\left|\psi\right\rangle_\eta = \alpha_1 \cdot \left|A_1\right\rangle_\eta + \ldots + \alpha_N \cdot \left|A_N\right\rangle_\eta, \tag{12}$$

where η means the η-th quantum object.

If we take into consideration the window of i quantum objects, the corresponding wave function $\left|\widetilde{\psi}\right\rangle_\eta$ is given by the Kronecker Product [14]:

$$\left|\widetilde{\psi}\right\rangle_\eta = \left|\psi\right\rangle_{\eta+1} \otimes \left|\psi\right\rangle_{\eta+2} \otimes \ldots \otimes \left|\psi\right\rangle_{\eta+i}, \tag{13}$$

where $\left\|\widetilde{\psi}\right\rangle_\eta\Big|^2$ give us probabilities that measurements on the set of i quantum objects $\{\eta + 1, \ldots, \eta + i\}$ will yield into a series of predefined events.

Now we assume that we have the time series, and by return, we estimate the wave functions (4). The algorithm for estimating the parameters of the wave functions can be decomposed into following steps:

1. Let us start with two events $\{A_1, A_2\}$ and assume $\hat{v}_1 = 0$.

2. We can estimate the phase $\hat{v}_2$ from the following equation:

$$\begin{aligned} &\mathrm{P}(|\mathrm{A}_1\rangle_{\eta+1} \cup |\mathrm{A}_2\rangle_{\eta+2}) = |\psi(\mathrm{A}_1) + \psi(\mathrm{A}_2)|^2 \\ &= \hat{\alpha}_1^2 + \hat{\alpha}_2^2 - 2\hat{\alpha}_1 \cdot \hat{\alpha}_2 \cdot \cos(\hat{v}_2 - \hat{v}_1) = \hat{\alpha}_{1,2}^2 \end{aligned} \tag{14}$$

with the help of occurrence rates $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_{1,2}, \hat{v}_1$ estimated from time series.

In principle we estimate the probabilities: $\mathrm{P}(|\mathrm{A}_1\rangle_\eta), \mathrm{P}(|\mathrm{A}_2\rangle_\eta), \mathrm{P}(|\mathrm{A}_1\rangle_\eta \cup |\mathrm{A}_2\rangle_{\eta+1})$ and compute the parameters $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_{1,2}$.

3. We continue and extend the algorithm for three events $\{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3\}$

$$\begin{aligned} &\mathrm{P}(|\mathrm{A}_1\rangle_{\eta+1} \cup |\mathrm{A}_2\rangle_{\eta+2} \cup |\mathrm{A}_3\rangle_{\eta+3}) \\ &= |\psi(\mathrm{A}_1) + \psi(\mathrm{A}_2) + \psi(\mathrm{A}_3)|^2 \\ &= \hat{\alpha}_1^2 + \hat{\alpha}_2^2 + \hat{\alpha}_3^2 - 2\hat{\alpha}_1 \cdot \hat{\alpha}_2 \cdot \cos(\hat{v}_2 - \hat{v}_1) \\ &\quad -2\hat{\alpha}_2 \cdot \hat{\alpha}_3 \cdot \cos(\hat{v}_3 - \hat{v}_2) - 2\hat{\alpha}_1 \cdot \hat{\alpha}_3 \cdot \cos(\hat{v}_3 - \hat{v}_1), \end{aligned} \tag{15}$$

where the estimate of angle $\hat{v}_3$ can be computed from the following equation:

$$\begin{aligned} \hat{\alpha}_{1,2} \cdot \cos(\hat{v}_3 - \hat{v}_{1,2}) &= \hat{\alpha}_2 \cdot \cos(\hat{v}_3 - \hat{v}_2) \\ &\quad + \hat{\alpha}_1 \cdot \cos(\hat{v}_3 - \hat{v}_1) \end{aligned} \tag{16}$$

under knowledge of the estimated parameters $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_{1,2}$, $\hat{v}_1, \hat{v}_2, \hat{v}_{1,2}$. This step must be made numerically because equation (16) is non-linear.

4. The above described procedure can be extended to a general N-step, where the unknown angle $\hat{v}_N$ is assumed to be numerically computed from the following equation:

$$\begin{aligned} &\hat{\alpha}_{1,2,\ldots,N-1} \cdot \cos(\hat{v}_N - \hat{v}_{1,2,\ldots,N-1}) \\ &= \hat{\alpha}_{N-1} \cdot \cos(\hat{v}_N - \hat{v}_{N-1}) + \hat{\alpha}_{N-2} \cdot \cos(\hat{v}_N - \hat{v}_{N-2}) \\ &+ \ldots + \hat{\alpha}_2 \cdot \cos(\hat{v}_N - \hat{v}_2) + \hat{\alpha}_1 \cdot \cos(\hat{v}_N - \hat{v}_1) \end{aligned} \tag{17}$$

under knowledge of the estimated parameters $\hat{\alpha}_1, \ldots, \hat{\alpha}_{N-1}, \hat{\alpha}_{1,2,\ldots,N-1}, \hat{v}_1, \hat{v}_1, \ldots, \hat{v}_{N-1}, \hat{v}_{1,2,\ldots,N-1}$.

Fortunately, for binary time series we need only step 1 and 2, and the result can be given analytically.

5 Wave probabilistic model of non-ergodic binary time series

We start our discussion with probabilistic binary time series, and we will show models of the occurrence of zero or one (probability and structure) in the form of wave probabilistic functions.

Let us define the binary quantum object in "bra-ket" form [17]:

$$|\psi\rangle_\eta = \sqrt{1-p} \cdot |0\rangle_\eta + \sqrt{p} \cdot \mathrm{e}^{j\varphi} \cdot |1\rangle_\eta. \tag{18}$$

Parameter p defines the probability of occurrence of state $|1\rangle_\eta$ of time series position η. The probability of occurrence of a state $|0\rangle_\eta$ must be $(1-p)$. Phase φ plays the role of "structural" parameter that expresses the rate of time series randomness [16].

The ergodic theorem allows the time average of a conforming process to equal the ensemble average. In practice, this means that statistical sampling can be performed at one instant across a group of identical processes or sampled over time on a single process with no change in the measured result.

Quantum objects defined in (18) fulfill the ergodic theorem, because the time average of a series along the time trajectories exists almost everywhere and is related to the space (set of realizations) average:

$$\mathrm{E}_\psi \equiv \int_\Omega |\psi, t\rangle_\eta \, \mathrm{d}\,\rho(\eta) \equiv \lim_{n\to\infty} \frac{|\psi, t_1\rangle_\eta + \ldots + |\psi, t_n\rangle_\eta}{n}, \tag{19}$$

where E_ψ is the mean value of a stochastic process and $\rho(\eta)$ is a probability measure. The first part of (19) is the η-ensemble average, which does not depend on time t and so the process is stationary. The second part of (19) is the time-average of quantum objects $|\psi, t_1\rangle_\eta + \ldots + |\psi, t_n\rangle_\eta$ with respect to selected process realization η.

If the complex parameters of the "bra-ket" model are time dependent we speak of a non-ergodic probabilistic binary process. Since non-ergodic processes are very difficult to model we will ease our requirements only to a special class of quasi-non-ergodic processes. Quasi-non-ergodic processes are characterized by linear time invariant (LTI) evolution of complex wave functions $\psi(\mathrm{A}_i, t)$ assigned into states $|\mathrm{A}_i\rangle_\eta$ in time interval t and position η.

If we add the time varying phase parameter into a binary process we can rewrite "bra-ket" objects (18) as follows:

$$\begin{aligned} |\psi, t\rangle_\eta &= \sqrt{1-\mathrm{A}_\omega} \cdot |0\rangle_\eta + \sqrt{\mathrm{A}_\omega} \cdot \mathrm{e}^{j\varphi} \cdot \mathrm{e}^{j\omega t} \cdot |1\rangle_\eta \\ &= \sqrt{1-\mathrm{A}_\omega} \cdot |0\rangle_\eta + \sqrt{\mathrm{A}_\omega} \cdot \mathrm{e}^{j(\omega t + \varphi_\omega)} \cdot |1\rangle_\eta, \end{aligned} \tag{20}$$

where parameter A_ω defines the probability of occurrence of state $|1\rangle_\eta$. Parameter φ_ω expresses the initial structure of the studied set of η-positioned states. Parameter ω represents the frequency of continual structuring and randomizing of η positioned states. Due to the time evolution of complex parameters (20) the ergodic condition is not fulfilled.

The parameters for state $|0\rangle_\eta$ can easily be computed from the parameters assigned to state $|1\rangle_\eta$ because A_ω must be maximally equal to one to be a probability function and the phase assigned to state $|0\rangle_\eta$ is assumed to be the normalized reference phase presented in Theorem 1 and equal to zero. Frequency ω can be interpreted as the energy spent to "structure" or "randomize" the set of η-positioned states with respect to chosen frequency ω.

We use the notation A_ω, φ_ω as modulus and initial phase parameters assigned to frequency ω. They represent the frequency decomposition in the same way as the Fourier transform. Equation (20) can be used as one frequency com-

ponent (modulus and initial phase) of a non-ergodic binary quantum object.

The general periodic non-ergodic behavior can be expressed as the sum of different frequency components:

$$\begin{aligned}|\psi,t\rangle_\eta &= \psi(0,t)\cdot|0\rangle_\eta + \psi(1,t)\cdot|1\rangle_\eta \\ &= \sqrt{1-\left(\sum_{i=1}^{N}\sqrt{\mathrm{A}_{\omega_i}}\cdot \mathrm{e}^{j(\omega_i t+\varphi_i)}\right)^2}\cdot|0\rangle_\eta \\ &+\left(\sum_{i=1}^{N}\sqrt{\mathrm{A}_{\omega_i}}\cdot \mathrm{e}^{j(\omega_i t+\varphi_i)}\right)\cdot|1\rangle_\eta,\end{aligned} \tag{21}$$

where the final modulus and phase assigned to state $|1\rangle_\eta$ are:

$$\sum_{i=1}^{N}\sqrt{\mathrm{A}_{\omega_i}}\cdot \mathrm{e}^{j(\omega_i t+\varphi_i)} = \sqrt{\widetilde{\mathrm{A}}(t)}\cdot \mathrm{e}^{j\cdot\widetilde{\varphi}(t)}, \tag{22}$$

where $\widetilde{\mathrm{A}}(t)$ is the time evolution of the probability of state $|1\rangle_\eta$, and $\widetilde{\varphi}(t)$ is the evolution of the link among different η positioned quantum objects (expressing structuring and randomizing). The complex parameter assigned to $|0\rangle_\eta$ is computed from the normalization and reference condition:

$$|\psi,t\rangle_\eta = \sqrt{1-\widetilde{\mathrm{A}}(t)}\cdot|0\rangle_\eta + \sqrt{\widetilde{\mathrm{A}}(t)}\cdot \mathrm{e}^{j\cdot\widetilde{\varphi}(t)}\cdot|1\rangle_\eta. \tag{23}$$

Equation (23) can be understood as the "bra-ket" representation of general non-ergodic binary quantum objects.

We can see in (21) that for every state $|0\rangle_\eta, |1\rangle_\eta$ the discrete modules and phase spectrum can be defined. This means that the time-evolution is modeled by a periodic function. The discrete spectrum can be replaced by a continuous spectrum. In this case, the sums in (21) are replaced by integrals. This replacement means the transition from a Fourier series into the Fourier transform.

6 Illustrative example-binary time series

Let us take two values $\{\mathrm{A}_0 = 0, \mathrm{A}_1 = 1\}$ time series represented by two complex wave functions $\psi(\mathrm{A}_0)$ and $\psi(\mathrm{A}_1)$. In the next part we will use for simplicity the notation $\psi(\mathrm{A}_0) = \psi_0$ and $\psi(\mathrm{A}_1) = \psi_1$.

A. Complementarity principle

We demonstrate the complementarity principle of binary time series.

Discrete Fourier Transform (DFT) is defined for k, $i \in \{0, 1, \ldots, N-1\}$:

$$\mathrm{X}_k = \sum_{i=0}^{N-1} x_i \cdot \mathrm{e}^{-j\cdot i\cdot k\frac{2\pi}{N}},$$

$$x_i = \frac{1}{N}\sum_{k=0}^{N-1}\mathrm{X}_k \cdot \mathrm{e}^{j\cdot i\cdot k\frac{2\pi}{N}}.$$

Probability in x-representation is given from (4):

$$\begin{aligned}\mathrm{P}(\mathrm{A}_0 = 0) &= \psi_0\cdot\psi_0^* \\ \mathrm{P}(\mathrm{A}_1 = 1) &= \psi_1\cdot\psi_1^*\end{aligned} \tag{24}$$

First we compute DFT of wave functions ψ_0, ψ_1:

$$\begin{aligned}\overline{\psi}(k=0) &= \sum_{i=0}^{1}\psi_i\cdot \mathrm{e}^0 = \psi_0 + \psi_1, \\ \overline{\psi}(k=1) &= \sum_{i=0}^{1}\psi_i\cdot \mathrm{e}^{-j\cdot i\cdot\pi} = \psi_0 - \psi_1,\end{aligned} \tag{25}$$

where $\overline{\psi}(.)$ are DFT transformed functions.

The probability function in k-representation can be given [10]:

$$\begin{aligned}\widetilde{\mathrm{P}}(k=0) &= \widetilde{\psi}(k=0)\cdot\widetilde{\psi}^*(k=0) \\ &= |\psi_0|^2 + |\psi_1|^2 + 2|\psi_0\cdot\psi_1|\cdot\cos(\varphi),\end{aligned} \tag{26}$$

$$\begin{aligned}\widetilde{\mathrm{P}}(k=1) &= \widetilde{\psi}(k=1)\cdot\widetilde{\psi}^*(k=1) \\ &= |\psi_0|^2 + |\psi_1|^2 - 2|\psi_0\cdot\psi_1|\cdot\cos(\varphi),\end{aligned} \tag{27}$$

where φ means the phase difference between complex numbers ψ_0 and ψ_1.

The Inverse DFT of probabilities (26) and (27) yields into convolution in x-representation and describes the links between two successive quantum objects.

If the phase parameter is zero $\varphi = 0$ the values are strongly independent and the time series is fully random. Phase $\varphi = \pi/2$ explains that the probability of changes $\{0, 1\}$ or $\{1, 0\}$ in the time series is very low. A binary time series looks like $\{0, 0, 0, 0 \ldots 0, 0, 0, 1, 1, 1 \ldots 1, 1, 1, 1\}$. On the other hand if the phase parameter is $\varphi = \pi$ the probability of finding a pair $\{1, 1\}$ or $\{0, 0\}$ limits to zero. The corresponding time series looks like $\{0, 1, 0, 1, \ldots, 0, 1, 0, 1\}$ and seems to be fully deterministic.

B. Entanglement principle

Let us take two quantum objects $\left|\widetilde{\psi}\right\rangle_\eta$ of a time series with wave functions ψ_0 and ψ_1 defined in (13) and (18). Let us define the probability:

$$\begin{aligned}&\mathrm{P}((\mathrm{A}_\eta = 0)\cup(\mathrm{A}_{\eta+1} = 1) \\ &= |\psi_0|^2 + |\psi_1|^2 + 2\cdot|\psi_0\cdot\psi_1|\cdot\cos(\varphi),\end{aligned} \tag{28}$$

where φ is the phase difference between wave functions ψ_0 and ψ_1. Suppose now with respect to Theorem 3 that:

$$\mathrm{P}((\mathrm{A}_\eta = 0)\cup(\mathrm{A}_{\eta+1} = 1) = 0. \tag{29}$$

This case can occur for the following values of φ:

$$\varphi = a\cos\left(-\frac{1}{2}\frac{|\psi_0|^2 + |\psi_1|^2}{|\psi_0\cdot\psi_1|}\right). \tag{30}$$

If, for example, $\psi_0 = \psi_1 = \frac{1}{\sqrt{2}}$ then $\varphi = \pi$ represents the entanglement.

As the result of entanglement we can write that the following events will surely happen (there are no random values):

$$\mathrm{P}((\mathrm{A}_\eta = 1)\cap(\mathrm{A}_{\eta+1} = 0) = 1. \tag{31}$$

We can also start with the following probability, instead of (29):

$$\mathrm{P}((\mathrm{A}_\eta = 1) \cup (\mathrm{A}_{\eta+1} = 0) = 0. \quad (32)$$

Then the entanglement yields into:

$$\mathrm{P}((\mathrm{A}_\eta = 0) \cap (\mathrm{A}_{\eta+1} = 1) = 1. \quad (33)$$

Equations (32) and (33) can both be written to quantum "bra-ket" representation:

$$\left|\widetilde{\psi}\right\rangle_\eta = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \quad (34)$$

Measuring the first quantum object from representation $\left|\widetilde{\psi}\right\rangle_\eta$ (the probability of measuring event 0 is 1/2 and the probability of measuring 1 is also 1/2) fully determines the value which will be measured on the second object. Equation (34) is the well-known Bell state, which is used in many applications, e.g., in quantum teleportation, quantum cryptography, etc.

C. Wave function estimation

Let us define the time series:

$$\{0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0\}$$

with the estimated occurrence rates:

$$\hat{\mathrm{P}}(\mathrm{A}_\eta = 1) = \frac{7}{15},\ \hat{\mathrm{P}}(\mathrm{A}_\eta = 0) = \frac{8}{15} \quad (35)$$

$$\hat{\mathrm{P}}(\mathrm{A}_\eta = 0 \cap \mathrm{A}_{\eta+1} = 1) = \frac{3}{14}. \quad (36)$$

By using the inclusion-exclusion rule we can apply the estimated probabilities and write the following equation:

$$\begin{aligned}&\hat{\mathrm{P}}((\mathrm{A}_\eta = 0) \cup (\mathrm{A}_{\eta+1} = 1))\\ &= \hat{\mathrm{P}}(\mathrm{A}_\eta = 0) + \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)\\ &\quad -2\sqrt{\hat{\mathrm{P}}(\mathrm{A}_\eta = 0) \cdot \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)} \cdot \cos(\varphi)\\ &= \hat{\mathrm{P}}(\mathrm{A}_\eta = 0) + \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)\\ &\quad -\hat{\mathrm{P}}(\mathrm{A}_\eta = 0 \cap \mathrm{A}_{\eta+1} = 1)\end{aligned} \quad (37)$$

from which the estimated phase parameter can be computed:

$$\hat{\varphi} = \arccos\left(\frac{\hat{\mathrm{P}}(\mathrm{A}_\eta = 0 \cap \mathrm{A}_{\eta+1} = 1)}{2\sqrt{\hat{\mathrm{P}}(\mathrm{A}_\eta = 0) \cdot \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)}}\right) = 1.3543. \quad (38)$$

If we use the estimated angle we can compute the inclusion-exclusion probability as:

$$\begin{aligned}&\hat{\mathrm{P}}((\mathrm{A}_\eta = 0) \cup (\mathrm{A}_{\eta+1} = 1))\\ &= \hat{\mathrm{P}}(\mathrm{A}_\eta = 0) + \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)\\ &\quad -2\sqrt{\hat{\mathrm{P}}(\mathrm{A}_\eta = 0) \cdot \hat{\mathrm{P}}(\mathrm{A}_{\eta+1} = 1)} \cdot \cos(\hat{\varphi}) = \frac{11}{14},\end{aligned} \quad (39)$$

where the estimate (39) corresponds to the occurrence rate directly estimated from the time series:

$$\hat{\mathrm{P}}((\mathrm{A}_\eta = 0) \cup (\mathrm{A}_{\eta+1} = 1)) = \frac{11}{14}. \quad (40)$$

We can see the compliance between (39) and (40).

7 Conclusion

Wave probabilistic models have been introduced and a mathematical comparison between usually used probabilistic models and wave probabilistic models has been presented.

Mathematical theory points to the applicability of wave probabilistic models and their special features. Quantum entanglement is explained as the consequence of phase parameters and it can be interpreted as the resonance principle of wave functions. The results of wave function resonance are fully deterministic, spatially distributed states with many properties.

The complementarity principle presents the studied time series in both *x*- and *k*-representation, where *x*-representation provides us with probabilities of occurrence of different events and *k*-representation caries information about the links between events and how the time series is structured.

The general estimation algorithm for phase parameters of wave probabilities was introduced and shown on an illustrative example – a binary time series. We can understand that this methodology yields into new models taking into account the structure of time series.

The application of the methodology presented here for non-ergodic time series modelling is also described and shown on binary time series. This opens new ways for modelling non-ergodic or quasi-non-ergodic processes of this kind.

The inspiration for the problem defined here came from quantum physics [1, 4, 6, 7]. The analogy with quantum mechanics seems very interesting and will inspire future work in statistical modelling area and wave probabilistic models.

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